A Memoir on Biquaternions.

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Clifford's "preliminary sketch of biquaternions" contains an outline of a calculus devised by him for the analytical treatment of the theory of screws. Besides this sketch there are four fragments dealing with the same subject. Clifford's object was, apparently, so to extend Hamilton's quaternion calculus that it might afford the same help in the study of the screw (that is, of the linear complex), as in its original form it affords in the study of the point and straight line. This end he attained by the invention of the *biquaternion.* Hamilton's biquaternion was a quantity of the form $q + \sqrt{-1}\, q'$, where q, q' are quaternions with real coefficients: modern analysis, however, considers all quantities as complex that are not expressly assumed to be real, and accordingly it is unnecessary to give a distinctive name to a quaternion with complex coefficients. Clifford's biquaternion is also a quantity of the form $q + \omega q'$, where q, q' are ordinary quaternions, but ω is no longer a scalar: it is an operator, commutative with all other operators, and such that its square is a scalar. This being so it appears that a *bivector* represents a motor (screw), and that a biquaternion represents the *quotient* of two motors, that is an operator which changes a given motor into another given motor.

In the first part of the "preliminary sketch" Clifford gives these definitions, and also defines the operator ω: the definition gives $\omega^2 = 0$: when we come to consider the biquaternion we are stopped by a difficulty which, we are told, can be explained by considering our geometry as a particular case of a geometry in which that difficulty does not occur. We now come to the second part of the paper: here we have, first of all, an explanation of the fundamental conceptions of the non-euclidean geometry: this is followed by a statement, for the most part without proof, of the fundamental theorems in the geometry and kinematics of elliptic space. The operator ω is next introduced by a definition giving $\omega^2 = 1$,

and then by the introduction of two new symbols we are enabled to write the biquaternion in a form which does not present the difficulty that stopped us in our consideration of parabolic geometry. The rest of the paper consists of investigations of some of the fundamental formulæ in the theory of elliptic space.

Of the fragments above referred to, two contain nothing that is not in the "preliminary sketch"; a third contains the beginning of an investigation of the motion of a rigid body in elliptic space: in this the ideas of the "sketch" are employed, and the velocity-system of the body is represented by a bivector. Lastly there is a fragment in which two problems of the theory of screws are considered: the first problem is that of finding the axis of a given screw, and this is completely solved; the second must, I think, have been the investigation of the cylindroid: all that is preserved is an expression for the axis of the sum of two screws whose axes intersect at right angles, and for the angular distance of this axis from the intersection of the other two.

There are, besides, a few notes dealing chiefly with the geometry of elliptic space.

In a paper "On the application of the Ausdehnungslehre and of Quaternions to the different kinds of uniform space" published in the Cambridge Philosophical Society's Transactions, Mr. Homersham Cox has added the value of ω^2 for hyperbolic space to what had been done by Clifford, but though his paper is interesting, it cannot, I think, be considered as containing any new development of Clifford's calculus.

I have considered the case of elliptic space in a paper, "On the Theory of Screws in Elliptic Space," published in the "Proceedings" of the London Mathematical Society. In this paper I solve the fundamental problems of the theory of screws, and prove most of Clifford's theorems: but the methods used are Grassmann's and not Clifford's. In the present paper I give what appears to me to be a tolerably complete development of Clifford's calculus, in the hope that, if it serves no other useful purpose, it may at least have some interest as a commentary on the "preliminary sketch."

Starting with the definition of a biquaternion I investigate the fundamental metric functions: I then consider the analytical problems answering to the elementary problems in the theory of screws, and give some formulæ relating to parallels. In all this I consider the general biquaternion, and there is no attempt at geometrical interpretation. The results of this first part are used in the second part, where they receive a geometrical interpretation.

It will be seen that I have found it necessary to introduce several new symbols: this is to be regretted, but it was quite unavoidable. One of these symbols is of such fundamental importance that it will be worth while to consider it here. I mean the e which occurs in all the formulæ when developed: this is defined by the equation $\omega^2 = e^2$: in elliptic space we have $e = 1$, in parabolic space $e = 0$, in hyperbolic space $e = \sqrt{-1}$: this quantity e is in fact the reciprocal of the radius of curvature of the space: it is the $1/k$ which occurs in Lobatchewsky's formulæ: it is only by the introduction of this scalar that the formulæ can become applicable to the three kinds of space.

Another change that I have made calls for some remark. I differ from Clifford in representing the point not by a vector, but by a biquaternion: the result is that the biquaternion represents all the *forms* that occur in a space of three dimensions as well as their quotients. Lastly, I remark that the operator ω is essentially a matrix.

PART I. BIQUATERNIONS.

1. *The Biquaternion.*

The whole theory of biquaternions depends upon the introduction of a symbol ω of which the geometrical meaning need not at present concern us: all that we require to know is that in all combinations it may be treated as if it were a mere scalar multiplier, and that its square is a scalar. This scalar I denote by e^2.

The biquaternion Q is defined by the equation

$$Q = q + \omega q',$$

where q, q' are ordinary quaternions.

It follows that $$\omega Q = e^2 q' + \omega q.$$

It is convenient to denote q, q' by functional symbols involving Q, and accordingly I write

$$q = \mho Q$$
$$q' = \Omega Q,$$

so that $$Q = \mho Q + \omega \Omega Q,$$

and then $$\omega Q = e^2 \Omega Q + \omega \mho Q.$$

We can therefore say that $\mho(\omega Q) = e^2 \Omega Q$

$$\Omega(\omega Q) = \mho Q.$$

Now take two biquaternions $Q = q + \omega q'$

$$R = r + \omega r'.$$

Then $$QR = (qr + e^2 q' r') + \omega(qr' + q'r).$$

Therefore
$$\mho(QR) = \mho Q.\mho R + e^2 \Omega Q.\Omega R$$
$$\Omega(QR) = \mho Q.\Omega R + \Omega Q.\mho R.$$
These equations are of fundamental importance.

Let $A = a + \omega b$ be a *biscalar*: then
$$(a + \omega b)(a - \omega b) = a^2 - e^2 b^2,$$
therefore
$$a + \omega b . \frac{a - \omega b}{a^2 - e^2 b^2} = 1.$$
Therefore
$$(a + \omega b)^{-1} = \frac{a - \omega b}{a^2 - e^2 b^2}.$$

2. *Distances.*

In all that follows I use Hamilton's symbols SVK in their usual sense. I also use his N to denote QKQ. T is not used here to denote the square root of this, and is defined below.

We obviously have $\mathrm{X}Q = \mathrm{X}\mho Q + \omega \mathrm{X}\Omega Q$, or
$$\mho \mathrm{X} = \mathrm{X}\mho$$
$$\Omega \mathrm{X} = \mathrm{X}\Omega$$
if χ denotes S, V or K.

I now define all the metric functions used in this paper.

Q being any biquaternion I write
$$TQ = +\sqrt{\mho NQ} = +\sqrt{\mho(QKQ)}$$
so that
$$T^2 Q = \mho Q \mho KQ + e^2 \Omega Q \Omega KQ$$
$$= \mho QK(\mho Q) + e^2 \Omega QK(\Omega Q)$$
$$= N(\mho Q) + e^2 N(\Omega Q).$$
Two biquaternions Q, R determine three angles defined as follows:
$$\cos\ (QR) = \mho SQKR\ (\div)$$
$$\sin e\,[QR] = e\Omega SQKR\ (\div)$$
$$\sin\ \{QR\} = TVQKR\ (\div)$$
Divisor $= TQ.TR$ in each case.

I write $[Q]$ for $[Q^2]$: we have by definition
$$NQ = T^2 Q\big(1 + \omega e^{-1} \sin e\,[Q]\big).$$
We have $NV(QKR) = NQNR - S^2 QKR.$

Therefore $\mho NV(QKR) = \mho(NQNR) - \mho S^2(QKR)$
$$= \mho NQ.\mho NR + e^2 \Omega NQ \Omega NR - (\mho SQKR)^2 - e^2(\Omega SQKR)^2,$$
or $T^2 Q T^2 R \sin^2\{QR\} = \big(1 + \sin e\,[Q] \sin e\,[R] - \cos^2(QR) - \sin^2 e\,[QR]\big) T^2 Q T^2 R.$

Therefore $\sin^2\{QR\} = 1 + \sin e\,[Q] \sin e\,[R] - \cos^2(QR) - \sin^2 e\,[QR].$

If either $[Q]$ or $[R]$ vanishes, and $[QR]$ also vanishes, we get

$$\sin^2\{QR\} = 1 - \cos^2(QR)$$
$$\{QR\} = (QR).$$

I now define as follows:

(QR) is the *angle* of Q, R.
$[QR]$ is the *moment* of Q, R.
$\{QR\}$ is the *distance* of Q, R.
$[Q]$ is the *pitch* of Q.

3. *Axes.*

A biquaternion of zero pitch is called a *special* biquaternion: that is, R is a special biquaternion if $\qquad \mho NR = 0.$
Let Q be any biquaternion: then a special biquaternion R such that $Q = AR$, where A is a biscalar is called an axis of Q: it will be seen that the determination of the axis leads to a quadratic equation, so that in general a biquaternion has two axes, and it appears that they are of the form R, ωR.
We have $\qquad R = A^{-1}Q$
and A must be determined so that

$$\Omega NR = 0.$$

That is $\qquad \Omega N(A^{-1}Q) = 0$
or $\qquad \Omega(A^{-2}NQ) = 0.$
Therefore $\qquad \Omega A^{-2}\mho NQ + \mho A^{-2}\Omega NQ = 0.$
But if $\qquad A = a + \omega b$

$$A^{-1} = \frac{a - \omega b}{a^2 - e^2 b^2}$$

$$A^{-2} = \frac{a^2 + e^2 b^2 - 2\omega ab}{(a^2 - e^2 b^2)^2}.$$

Therefore we have $\qquad 0 = (a^2 + e^2 b^2)\Omega NQ - 2ab\mho NQ$
or

$$\frac{2ab}{a^2 + e^2 b^2} = \frac{\Omega NQ}{\mho NQ}$$
$$= \frac{1}{e}\sin e[Q]$$

or

$$\frac{2eab}{a^2 + e^2 b^2} = \sin e[Q].$$

This is obviously satisfied by

$$a = \cos e\frac{[Q]}{2}$$
$$b = e^{-1}\sin e\frac{[Q]}{2}.$$

And here I stop to make a remark of some importance: *all biquaternions are only determined to a scalar factor près*, and therefore we can take the values just written as the actual values of a, b. I now introduce two symbols which are constantly used in the sequel, viz. I write

$$\text{es}\,\phi = e^{-1}\sin e\phi$$
$$\text{ec}\,\phi = \cos e\phi.$$

I also write ϕ for $[Q]$ the pitch of Q: we therefore have

$$a = \text{ec}\,\frac{\varphi}{2}$$

$$b = \text{es}\,\frac{\varphi}{2}.$$

And then
$$a^2 - e^2 b^2 = \text{ec}\,\phi.$$

We therefore get
$$Q = \left(\text{ec}\,\frac{\varphi}{2} + \omega\,\text{es}\,\frac{\varphi}{2}\right) R,$$

$$\text{ec}\,\phi R = \left(\text{ec}\,\frac{\varphi}{2} - \omega\,\text{es}\,\frac{\varphi}{2}\right) Q.$$

This gives
$$\text{ec}\,\phi . \omega R = \left(- e^2\,\text{es}\,\frac{\varphi}{2} + \omega\,\text{ec}\,\frac{\varphi}{2}\right) Q,$$

and it can be immediately verified that

$$- e^2\,\text{es}\,\frac{\varphi}{2}, \quad -\text{ec}\,\frac{\varphi}{2}$$

are the other roots of the equation giving $\frac{a}{b}$, and therefore, as was stated above, the two axes of Q are of the form R, ωR.

There is an important case in which the investigation fails, viz. the case in which $\text{es}\,\phi = \pm e^{-1}$ or $e\phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$: we have $\text{ec}\,\phi = 0$, and the expression for the axis becomes infinite: the fact is that, as will be seen below, the axis is really indeterminate.

Let Q, Q' be two biquaternions; R, R' two of their axes, and ϕ, ϕ' their pitches.

Let
$$Q = AR$$
$$Q' = A'R'$$

we have
$$A = \text{ec}\,\frac{\varphi}{2} + \omega\,\text{es}\,\frac{\varphi}{2}$$

$$A' = \text{ec}\,\frac{\varphi'}{2} + \omega\,\text{es}\,\frac{\varphi'}{2}.$$

Therefore
$$A^2 = 1 + \omega \text{ es } \phi$$
$$AA' = \text{ec}\,\frac{\varphi - \varphi'}{2} + \omega \text{ es } \frac{\varphi + \varphi'}{2}$$
$$NQ = A^2 NR.$$
Therefore
$$T^2 Q = \mho A^2 \mho NR$$
$$= \mho NR$$
$$= T^2 R$$
since $\Omega NR = 0$, by definition moreover
$$QKQ' = AA'.\,RKR'.$$
Therefore
$$\mho SQKQ' = \mho AA'.\mho SRKR' + e^2 \Omega AA'.\Omega SRKR'$$
or
$$\cos(QQ') = \text{ec}\,\frac{\varphi - \varphi'}{2}.\cos(RR') + e^2.\text{ es } \frac{\varphi + \varphi'}{2}.\text{ es }[RR'].$$
We have also
$$\Omega SQKQ' = \mho AA'.\Omega SRKR' + \Omega AA' \mho SRKR'$$
or
$$\sin[QQ'] = \text{ec}\,\frac{\varphi - \varphi'}{2} \text{ es }[RR'] + \text{es } \frac{\varphi + \varphi'}{2} \cos(RR')$$

4. *Vectors.*

I now introduce two operators, ξ, η, defined as follows:
$$\xi = \frac{1 + e^{-1}\omega}{2}$$
$$\eta = \frac{1 - e^{-1}\omega}{2}.$$
It is important to notice that, like ω, ξ and η can be treated as if they were mere scalar multipliers.

We have
$$\xi^2 = \frac{1 + 2e^{-1}\omega + e^{-2}\omega^2}{4}$$
$$= \frac{1 + 2e^{-1}\omega + 1}{4}$$
$$= \frac{1 + e^{-1}\omega}{2}$$
$$= \xi.$$
Similarly
$$\eta^2 = \eta.$$
Moreover
$$\xi\eta = \frac{(1 + e^{-1}\omega)(1 - e^{-1}\omega)}{4}$$
$$= \frac{1 - e^{-2}\omega^2}{4}$$
$$= 0.$$

If $\xi Q = 0$, Q is called a ξ-vector; if $\eta Q = 0$, Q is called an η-vector, as the word *vector* when used in this sense is always accompanied by a reference to the two species of vectors there need be no confusion between this kind of vector and the ordinary vector of quaternions.

We have
$$\xi Q = \frac{1 + e^{-1}\omega}{2} Q.$$
Therefore
$$\Omega \xi Q = \frac{\Omega Q + e^{-1}\mho Q}{2}$$
$$\Omega \xi Q = \frac{\mho Q + e\Omega Q}{2}.$$

If Q is a ξ-vector we must have $\Omega \xi Q = \mho \xi Q = 0$: these equations agree in giving
$$\Omega Q = - e^{-1} \mho Q.$$
Therefore
$$Q = (1 - e^{-1}\omega) \mho Q$$
$$= 2\eta \mho Q.$$
Therefore any ξ-vector can be written in the form ηQ: and in fact it is obvious that ηQ is a ξ-vector, because $\xi\eta = 0$: in the same way any η-vector can be written in the form ξQ. It is as well to notice that in these expressions, ξQ, ηQ, Q is a simple quaternion, not a biquaternion, and that, if $R = \xi Q$ is an η-vector, we have
$$Q = 2\mho R,$$
and this is of course also true if $R = \eta Q$ is a ξ-vector.

Any biquaternion can be written, and in one way only, as the sum of a ξ-vector and an η-vector. Let Q be a given biquaternion, then if we are to have
$$Q = R + S$$
where R is a ξ-vector, and S an η-vector, we have also
$$e^{-1}\omega Q = e^{-1}\omega R + e^{-1}\omega S$$
$$= - R + S,$$
by definition: therefore
$$S = \frac{Q + e^{-1}\omega Q}{2}$$
$$= \xi Q$$
and
$$R = \eta Q.$$
Therefore
$$Q = \eta Q + \xi Q$$
is the only decomposition of a biquaternion into a ξ-vector and an η-vector: it is obvious *a priori* that this is such decomposition, and what we have proved is that there is no other.

At the end of (4) it was stated that if $[Q] = \frac{\pi}{2e}$ or $\frac{3\pi}{2e}$ we cannot find the axes of Q: I show now that this happens if Q is an η-vector or a ξ-vector.

Let ξq be any η-vector: q being a simple quaternion.

We have
$$N(\xi q) = \xi q . K\xi q$$
$$= \xi q K q$$
$$= \xi (Nq).$$

Now $\Omega NQ = 0$ because Q is a simple quaternion.

Therefore
$$\mho (N\xi q) = \frac{Nq}{2}$$
$$\Omega (N\xi q) = \frac{e^{-1}Nq}{2}.$$

Therefore
$$\text{es } [\xi q] = e^{-1}$$
$$\sin e \, [\xi q] = 1$$
$$[\xi q] = \frac{\pi}{2e}.$$

In the same way we get
$$[\eta q] = \frac{3\pi}{2e}.$$

It should be noticed that if $Q = \xi q$ is an η-vector we have $\xi Q = Q$, and in the same way if $Q = \eta q$ is a ξ-vector we have $\eta Q = Q$; and, conversely, it is obvious that these equations $Q = \xi Q$, $Q = \eta Q$ define Q as an η-vector or a ξ-vector respectively.

Let ξq, ηr be two vectors of different species: we have
$$\xi q . \eta r = \xi \eta . qr$$
$$= 0.$$

This of course breaks up into a scalar and a vector equation: and we can say that if Q, R are two vectors of different species we have
$$VQR = 0$$
$$SQR = 0.$$

Each of these equations breaks up into two, giving two vector equations and two scalar equations; the scalar equations give
$$(QR) = \frac{\pi}{2}$$
$$[QR] = 0.$$

5. *Parallels.*

Let Q, R be two biquaternions: then, if $\xi Q = \xi R$, Q, R are said to be ξ-parallel: if $\eta Q = \eta R$ they are said to be η-parallel. All ξ-vectors are ξ-parallel, and all η-vectors are η-parallel. If two biquaternions are both ξ (η) parallel to the same biquaternion they are ξ (η) parallel to each other. Since all biquater-

nions are only determined to a scalar factor *près*, so that λR is the same as R if λ is a simple scalar, we can say that QR are parallel if $\xi Q = \lambda\xi R$, or if $\eta Q = \lambda\eta R$.

If Q is both ξ-parallel and η-parallel to R, Q is of the form $\lambda\xi R + \mu\eta R$: if we have $\xi Q = \xi R$, $\eta Q = \eta R$ we have

$$\begin{aligned} Q &= \xi Q + \eta Q \\ &= \xi R + \eta R \\ &= R \end{aligned}$$

and the two biquaternions are identical. Now let

$$\begin{aligned} \xi Q &= \lambda\xi R \\ \eta Q &= \mu\eta R. \end{aligned}$$

Then

$$\begin{aligned} Q &= \xi Q + \eta Q \\ &= \lambda\xi R + \mu\eta R. \end{aligned}$$

This gives

$$Q = \frac{\lambda + \mu}{2} R + \frac{e^{-1}(\lambda - \mu)}{2} \omega R.$$

Conversely if we have $\quad Q = aR + b\omega R$

where a and b are scalars, Q is ξ-parallel and η-parallel to R.

To prove this I observe that

$$\begin{aligned} \xi\omega &= \frac{1 + e^{-1}\omega}{2}\omega \\ &= \frac{\omega + e^{-1}\omega^2}{2} \\ &= \frac{\omega + e}{2} \\ &= e\xi. \end{aligned}$$

Similarly

$$\eta\omega = -e\eta.$$

Therefore if

$$\begin{aligned} Q &= aR + b\omega R \\ \xi Q &= a\xi R + b\xi\omega R \\ &= (a + be)\,\xi R \end{aligned}$$

and

$$\eta Q = (a - be)\,\eta R.$$

Therefore, as was stated above, Q is ξ-parallel and η-parallel to R. In particular, it follows that the axis of a biquaternion is ξ-parallel and η-parallel to the biquaternion.

Now let ξr be an η-vector, and Q any biquaternion: we have

$$\begin{aligned} Q\xi r &= (\xi Q + \eta Q)\,\xi r \\ &= \xi Q . \xi r. \end{aligned}$$

Similarly if Q' is any other biquaternion,

$$Q'\xi r = \xi Q' . \xi r.$$

Therefore if Q is ξ-parallel to Q' so that $\xi Q = \xi Q'$ we have

$$Q.\xi r = Q'.\xi r.$$

In the same way if Q is η-parallel to Q' we have

$$Q.\eta r = Q'.\eta r.$$

If we have $$\xi Q = \xi R$$
we get $$N\xi Q = \chi N(\xi R)..$$
But $$N\xi Q = \xi Q.\xi KQ$$
$$= \xi QKQ$$
$$= \xi NQ.$$
Therefore $$\mho(N\xi Q) = \frac{\varpi NQ + e\Omega NQ}{2}$$
$$\Omega(N\xi Q) = \frac{e^{-1}\varpi NQ + \Omega NQ}{2}.$$

Therefore we have

$$T^2 Q(1 + \sin e[Q]) = T^2 R(1 + \sin e[R]).$$

In the same way if $$\xi U = \xi W$$
$$T^2 U(1 + \sin e[U]) = T^2 W(1 + \sin e[W]).$$
Moreover $$\xi(QKU) = \xi(RKW).$$
Therefore
$$TUTQ(\cos(QU) + \sin e[QU]) = TRTW(\cos(RW) + \sin e[RW]).$$
Therefore, finally,
$$\frac{\cos(QU) + \sin e[QU]}{(1 + \sin e[Q])^{\frac{1}{2}}(1 + \sin e[U])^{\frac{1}{2}}} = \frac{\cos(RW) + \sin e[RW]}{(1 + \sin e[R])^{\frac{1}{2}}.(1 + \sin e[W])^{\frac{1}{2}}}.$$
If $$aQ + b\omega Q + cR + d\omega R = 0,$$
and Q, R, ωR are not connected by a linear relation, Q, R are either ξ-parallel or η-parallel: this can only be the case if $e^2 = 1$. For operating with ω on the equation $$aQ + b\omega Q + cR + d\omega R = 0$$
we get $$a\omega Q + e^2 bQ + c\omega R + e^2 dR = 0.$$

Substituting for ωQ in the first equation from the second we get

$$Q\left(a - \frac{e^2 b^2}{a}\right) + R\left(c - \frac{e^2 bd}{a}\right) + \omega R\left(d - \frac{e^2 bc}{a}\right) = 0.$$

Therefore, by what was stipulated as to Q, R, ωR,

$$a^2 = e^2 b^2$$
$$ac = e^2 bd$$
$$ad = e^2 bc.$$

The first equation gives $a = \pm eb$: substituting in the second and third we get either $a = 0$, $e = 0$ (this case will be considered below), or

$$c = \pm ed$$
$$d = \pm ec.$$

These equations are inconsistent unless $e^2 = 1$: assuming this to be the case, and substituting, we get

$$eb(1 \pm e^{-1}\omega)\,Q + ed(1 \pm e^{-1}\omega)\,R = 0.$$

That is
$$b \genfrac{}{}{0pt}{}{\xi}{\eta} Q + d \genfrac{}{}{0pt}{}{\xi}{\eta} R = 0.$$

That is Q, R are either ξ-parallel or η-parallel.

If we have $a = 0$, $e = 0$, we get

$$b\omega Q + cR + d\omega R = 0,$$

and then operating with ω we get $c = 0$, and the equation reduces to

$$b\omega Q + d\omega R = 0.$$

Now if $e = 0$, $2\xi Q = (1 + e^{-1}\omega)\,Q$ becomes $e^{-1}\omega Q$, and $2\eta Q$ becomes $-2e^{-1}\omega Q$: therefore the last equation is

$$b \genfrac{}{}{0pt}{}{\xi}{\eta} Q + d \genfrac{}{}{0pt}{}{\xi}{\eta} R.$$

Therefore the theorem holds in this case also, except that the two species of parallelism coincide.

In the other excluded case in which we have a relation

$$aQ + bR + c\omega R = 0$$

we get the case considered above in which Q, R are both ξ-parallel and η-parallel.

Therefore we can neglect all distinctions of cases and say generally that Q, R are parallel if there is any relation of the form

$$aQ + b\omega Q + cR + d\omega R = 0.$$

I now prove that, as was stated above, the axis of a vector of either species is indeterminate. Let ξq be a vector: R an axis: we are to have

$$\xi q = aR + b\omega R.$$

Therefore
$$0 = \xi\eta q = a\eta R - eb\eta R.$$

Therefore
$$a = eb.$$

Therefore
$$\xi q = a(1 + e^{-1}\omega)\,R.$$

Therefore R may be any special biquaternion satisfying this condition.

Therefore, the axis of an η-vector is any special biquaternion ξ-parallel to it, and the axis of a ξ-vector is any special biquaternion η-parallel to it.

6. *The Cylindroid.*

Let A, B be two biquaternions: they determine a linear singly infinite series of biquaternions $\lambda A + \mu B$ where λ, μ are scalars: this set is called a cylindroid, so that if C is any biquaternion of the cylindroid (A, B) we have

$$C = \lambda A + \mu B.$$

Every cylindroid contains two special biquaternions (v. def. in (3)): for we have

$$C = \lambda A + \mu B.$$

Therefore
$$KC = \lambda KA + \mu KB$$

$$NC = CKC = \lambda^2 NA + 2\lambda\mu SAKB + \mu^2 NB.$$

Therefore if C is to be a special biquaternion, so that $\Omega NC = 0$, we must have

$$\lambda^2 \Omega NA + 2\lambda\mu\Omega SAKB + \mu^2 \Omega NB = 0,$$

or
$$\frac{\lambda^2}{T^2B} \text{ es } [A] + \frac{2\lambda}{TB}\frac{\mu}{TA} \text{ es } [AB] + \frac{\mu^2}{T^2A} \text{ es } [B] = 0.$$

This equation determines the two values of $\frac{\lambda}{\mu}$ corresponding to the two special biquaternions: if A, B are special biquaternions and such that $[AB] = 0$, the equation is an identity: every biquaternion of the cylindroid is a special biquaternion, and it can be verified at once that we have also $[AC] = [BC] = 0$. The roots coincide if $[AB] = e^{-1}\sin^{-1} e$.

Every cylindroid contains in general two biquaternions C, C', satisfying the *two* conditions $SCKC' = 0$.

Let
$$C = \lambda A + \mu B$$
$$C' = \lambda' A + \mu' B.$$

Then
$$SCKC' = \lambda\lambda' NA + (\lambda\mu' + \lambda'\mu)\, SAKB + \mu\mu' NB.$$

But we are to have $\Omega SCKC' = \mho SCKC' = 0$: therefore

$$\lambda\lambda' \mho NA + (\lambda\mu' + \lambda'\mu)\, \mho SAKB + \mu\mu' \mho NB = 0$$
$$\lambda\lambda' \Omega NA + (\lambda\mu' + \lambda'\mu)\, \Omega SAKB + \mu\mu' \Omega NB = 0.$$

Therefore $\frac{\lambda}{\mu}$, $\frac{\lambda'}{\mu'}$ are the two roots of

$$\begin{vmatrix} 1 & -x & x^2 \\ \mho NA & \mho SAKB & \mho NB \\ \Omega NA & \Omega SAKB & \Omega NB \end{vmatrix} = 0.$$

Every cylindroid contains, in general, two biquaternions C, C' such that

$$TC = TC' = 0.$$

If
$$C = \lambda A + \mu B$$

we are to have
$$0 = \mho NC$$
$$= \lambda^2 \mho NA + 2\lambda\mu \mho SAKB + \mu^2 \mho NB.$$

This is an identity if $TA = TB = 0$, $(AB) = \frac{\pi}{2}$: the roots coincide if $(AB) = 0$.

If a cylindroid contains an infinite number of special biquaternions, or an infinite number of biquaternions whose tensor vanishes, it contains an infinite number of pairs such that $SCKC' = 0$. This is obvious from what precedes.

Let Q, R be any two biquaternions: we have

$$V(\lambda Q + \mu R)\, K(\lambda' Q + \mu' R) = (\lambda\mu' - \lambda'\mu)\; VQKR.$$

Therefore $VQKR$ is the same (to a scalar factor *près*) for every pair of biquaternions of the cylindroid (Q, R): this bivector $VQKR$ is called the axis of the cylindroid.

If $\xi Q = \xi R = U$ we have $\xi VQKR = V(\xi Q . \xi KR) = VUKU = 0$: therefore if two biquaternions are parallel, the axis of the cylindroid they determine is a vector of the same species as the parallelism. Moreover, if $P = \lambda Q + \mu R$ we have $\xi P = \lambda \xi Q + \mu \xi R = (\lambda + \mu)\, U$: therefore all the biquaternions of the cylindroid are parallel.

If $\xi A = \xi B$, $\xi C = \xi D$, we have

$$\xi (AKC) = \xi (BKD),$$

and therefore

$$\xi V(AKC) = \xi V(BKD).$$

Now if E, F are two biquaternions of the cylindroids (A, C), (B, D) respectively, and such that

$$E = \lambda A + \mu C$$
$$F = \lambda B + \mu D.$$

We have

$$\begin{aligned} \xi E &= \lambda \xi A + \mu \xi C \\ &= \lambda \xi B + \mu \xi D \\ &= \xi F. \end{aligned}$$

Therefore to every biquaternion of one cylindroid corresponds a parallel biquaternion of the other cylindroid: we may therefore say that two parallel pairs of biquaternions determine two parallel cylindroids, and the axes of parallel cylindroids are parallel. In all that precedes ξ stands for ξ or η.

PART II. GEOMETRY.

In the first part we have considered the analytical theory of the biquaternion, apart from any interpretation. In the second part we make use of the results of the first part, and interpret them geometrically. The greater part of the first two sections of Part II is foreign to the object of this paper, which is

the development of metric geometry from the definition of the biquaternion: but they will, I think, make the rest of the paper more easily intelligible.

For convenience the sections of the whole paper are numbered consecutively.

7. *The Absolute.*

Suppose the tetrahedron of reference self-conjugate with respect to the absolute; and suppose, moreover, that the equation of the absolute is

$$e^2(x^2+y^2+z^2)+\omega^2=0;$$

then the plane-equation is $l^2+m^2+n^2+e^2p^2=0$, and the line equation (the condition that a line may touch the surface is $f^2+g^2+h^2+e^2(a^2+b^2+c^2)=0$, we know that in elliptic space the absolute is $x^2+y^2+z^2+\omega^2=0$, and that in hyperbolic space it is $\omega^2-(x^2+y^2+z^2)=0$: in parabolic space the absolute becomes a plane conic (an infinitesimal quadric) in the plane at infinity, so that its point equation is $(\text{const.})^2=0$. if $(\text{const.})=0$ is the plane at infinity, and its plane equation is $l^2+m^2+n^2=0$: we therefore see that we can represent the three geometries by taking for the three equations of the absolute

$$e^2(x^2+y^2+z^2)+\omega^2=0$$
$$l^2+m^2+n^2+e^2p^2=0$$
$$f^2+g^2+h^2+e^2(a^2+b^2+c^2)=0$$

with the following stipulations.

I. In elliptic space $e^2=1$

In parabolic space $e^2=0$

In hyperbolic space $e^2=-1$.

II. In parabolic space $\delta=1$ for all points at a finite distance and $p=1$ for planes not passing through (0001).

III. In parabolic space a line is considered to touch the absolute if it meets the "circle at infinity": for the absolute consists of this curve taken twice over.

The generators of the absolute are the tangents of the "circle at infinity": each tangent represents two generators, one of each system.

If $(\alpha, \beta, \gamma, \delta)$ is any point, its polar plane with respect to the absolute is $(e^2\alpha, e^2\beta, e^2\gamma, \delta)$, and if $(abcfgh)$ is any line its polar is (fgh, e^2a, e^2b, e^2c): and we can say more generally that this is the polar of the screw $(abcfgh)$.

If a screw is its own conjugate we must have

$$a=\lambda f,\ b=\lambda g,\ c=\lambda h,\ f=e^2\lambda a,\ g=e^2\lambda b,\ h=e^2\lambda c.$$

This gives
$$\lambda^2e^2=1$$
$$\lambda=\pm e^{-1}.$$

Therefore if a screw is its own conjugate its coordinates are of the form

$$(\pm e^{-1}f,\ \pm e^{-1}g,\ \pm e^{-1}h,\ f,\ g,\ h).$$

If a line is its own conjugate it is a generator of the absolute and we get as the coordinates of a generator of one system

$$(e^{-1}f,\ e^{-1}g,\ e^{-1}h,\ f,\ g,\ h)$$

and as the coordinates of a generator of the other system

$$(-e^{-1}f,\ -e^{-1}g,\ -e^{-1}h,\ f,\ g,\ h).$$

8. *Interpretations.*

There is in three dimensional space a ∞^3 series of points, a ∞^3 series of planes, a ∞^5 series of screws (motors, linear complexes): the line is a particular case of the screw, and need not, at present, be considered separately. The biquaternion $q + \omega q'$ contains eight scalar constants, but it is determined (not but by their absolute values but) by their ratios: there is therefore a ∞^7 series of biquaternions. Therefore, there is a ∞^3 series satisfying four scalar conditions, and a ∞^5 series satisfying two conditions: therefore we can make a biquaternion satisfying four conditions represent a point or a plane, and a biquaternion satisfying two conditions represent a screw.

These conditions are chosen as follows: The biquaternion $q + \omega q'$ represents

$$\text{a } \textit{point} \text{ if } Vq = 0,\ Sq' = 0$$
$$\text{a } \textit{plane} \text{ if } Sq = 0,\ Vq' = 0$$
$$\text{a } \textit{screw} \text{ if } Sq = 0,\ Sq' = 0.$$

Moreover, if $(\alpha\beta\gamma\delta)$ are scalars, we say that

$$\delta + \omega(\alpha i + \beta j + \gamma k)$$
$$\alpha i + \beta j + \gamma k\, \omega\delta$$

represent the point and plane respectively whose coordinates in the system of (7) are $(\alpha\beta\gamma\delta)$.

Lastly the screw $(abcfgh)$ is represented by

$$(fi + gj + \gamma k) - \omega(ai + bj + \gamma k).$$

I justify this by showing, in a single instance, that these determinations give metric formulæ agreeing with those in (7): the representation of a screw will be justified by the expression for a line obtained below.

The angle and distance of two points are given by

$$\cos(PP') = \frac{\mho SPKP'}{TP.TP'},$$

$$\sin\{PP'\} = \frac{\Omega SPKP'}{TP.TP'}.$$

Now let

$$P = \delta + \omega\rho = \delta + \omega(\alpha i + \beta j + \gamma k), \quad P' = \delta' + \omega\rho' = \delta' + \omega(\alpha' i + \beta' j + \gamma' k),$$

we have

$$\begin{aligned} PKP' &= (\delta + \omega\rho)(\delta' + \omega\rho') \\ &= \delta\delta' - e^2\rho\rho' + \omega(\rho\delta' - \rho'\delta), \\ PKP &= \delta^2 - e^2\rho^2. \end{aligned}$$

Therefore

$$\begin{aligned} \cos(PP') &= \frac{\delta\delta' - e^2 S\rho\rho}{(\delta^2 - e^2\rho^2)^{\frac{1}{2}}.(\delta'^2 - e^2\rho'^2)^{\frac{1}{2}}} \\ &= \frac{\delta\delta' + e^2(\alpha\alpha' + \beta\beta' + \gamma\gamma')}{\{\delta^2 + e^2(\alpha^2 + \beta^2 + \gamma^2)\}^{\frac{1}{2}}.\{\delta'^2 + e^2(\alpha'^2 + \beta'^2 + \gamma'^2)\}^{\frac{1}{2}}}. \end{aligned}$$

But this is the known expression for $\cos PP'$ with $\delta^2 + e^2(\alpha^2 + \beta^2 + \gamma^2) = 0$ as the equation of the absolute.

Moreover, we get at once $[PP'] = 0$, and therefore by (1) $\{PP'\} = (PP')$.

Now, writing $(\alpha\beta\gamma)$ for $\alpha i + \beta j + \gamma k$, let

$$P = \delta + \omega(\alpha\beta\gamma)$$

be a point: then

$$\omega P = e^2(\alpha\beta\gamma) + \omega\delta.$$

But this is the plane $(e^2\alpha, e^2\beta, e^2\gamma, \delta)$ which is the polar plane of $(\alpha\beta\gamma\delta)$ with respect to the absolute: therefore ωP is the polar of P. In the same way if Q is a plane or a screw, we see that ωQ is the polar point or screw with respect to the absolute.

This is the geometrical interpretation of ω referred to in (1).

9. *Lines.*

If P, P' are two points, the line joining them is the axis of the cylindroid $(\omega P, P')$: for this is the same for all pairs $\lambda P + \mu P'$, and for such pairs only: but $\lambda P + \mu P'$ is a linear ∞^1 series of points containing P, P': therefore it is the points of the line (PP'): therefore $V(\omega P.KP')$ can be taken to represent the line.

We have, if $P = \delta + \omega\rho$, $P' = \delta' + \omega\rho'$

$$\begin{aligned} V(\omega P.KP') &= V(e^2\rho + \omega\delta)(\delta' - \omega\rho') \\ &= -e^2\omega V\rho\rho' + e^2(\rho\delta' - \rho'\delta), \end{aligned}$$

or, dividing by e^2, the line joining PP' is

$$\rho\delta' - \delta\rho' - \omega V\rho\rho'.$$

This is a bivector $(\alpha + \omega\alpha')$, and we can verify at once that $S\alpha\alpha' = 0$: therefore $\alpha + \omega\alpha'$ is a line only if $S\alpha\alpha' = 0$, that is $[\alpha + \omega\alpha'] = 0$; moreover it will appear below that if $S\alpha\alpha'$ vanishes, $\alpha + \omega\alpha'$ is always a line.

The conditions that a point may be on a line can be found as follows: Let $\delta + \omega\rho$, $\delta' + \omega\rho'$ be two points on the line $\alpha + \omega\alpha'$: then we must have

$$\rho\delta' - \delta\rho' = \alpha$$
$$V\rho\rho' = -\alpha'.$$

The required conditions are to be found by eliminating (ρ', δ'), the first equation gives
$$\rho' = \frac{\rho\delta' - \alpha}{\delta}.$$
Substituting in the second we get $\dfrac{-V\rho\alpha}{\delta} = -\alpha'$,

or
$$N\rho\alpha - \delta\alpha' = 0.$$

This gives three conditions: we know that this is one more than we require, but it is convenient to keep them all, and indeed to add a fourth, obtained by operating with $S.\rho$, we thus get as the complete set

$$V\rho\alpha - \delta\alpha' = 0$$
$$S\rho\alpha' = 0.$$

If we operate on the first with $S\alpha$ we get the known condition $S\alpha\alpha' = 0$.

The last condition gives $\rho = V\alpha'\varepsilon$,
where ε is some vector: substituting in the other we get

$$V\alpha V\alpha'\varepsilon + \delta\alpha' = 0$$

or
$$\varepsilon S\alpha\alpha' - \alpha'(S\alpha\varepsilon - \delta) = 0.$$

This equation is possible, if, and only if $S\alpha\alpha' = 0$, and then it gives $\delta = S\alpha\varepsilon$.

This proves that $\alpha + \omega\alpha'$ is a line if $S\alpha\alpha' = 0$, and that any point on the line can be represented by $S\alpha\varepsilon + \omega V\alpha'\varepsilon$.
Moreover it can be verified at once that the point $0 + \omega\alpha$ is always on the line.

To find the intersection of two concurrent lines, and the condition that two straight lines may intersect, let $\alpha + \omega\alpha'$, $\beta + \omega\beta'$ be the lines: $\delta + \omega\rho$ their intersection: we must have

$$S\rho\alpha' = 0$$
$$S\rho\beta' = 0$$
$$V\rho\alpha - \delta\alpha' = 0$$
$$V\rho\beta - \delta\beta' = 0.$$

The first two give $\rho = xV\alpha'\beta'$, and then

$$xV.\alpha V\alpha'\beta' + \delta\alpha' = 0$$
$$xV.\beta V\alpha'\beta' + \delta\beta' = 0$$

or
$$x(\beta' S\alpha\alpha' - \alpha' S\alpha\beta') + \delta\alpha' = 0$$
$$x(\beta' S\alpha'\beta - \alpha' S\beta\beta') + \delta\beta' = 0.$$

Now $S\alpha\alpha' = S\beta\beta' = 0$, and therefore these equations are consistent if, and only if

$$S\alpha\beta' + S\alpha'\beta = 0,$$

and then they agree in giving

$$\begin{aligned}\delta &= xS\alpha\beta' \\ &= -xS\alpha'\beta.\end{aligned}$$

Therefore we can take

$$\omega V\alpha'\beta' + S\alpha\beta'$$

for the intersection of $\alpha + \omega\alpha'$, $\beta + \omega\beta'$, and the condition of intersection is

$$S\alpha\beta' + S\alpha'\beta = 0.$$

It can be verified at once that this is

$$[(\alpha + \omega\alpha')(\beta + \omega\beta')] = 0.$$

Therefore if Q is a bivector

$$[Q] = 0$$

is the condition that it may represent a line

$$[QQ'] = 0$$

is the condition that Q, Q' may intersect.

In exactly the same way the conditions that a plane $\omega\delta + \rho$ may pass through a line are

$$S\rho\alpha = 0$$

$$V\rho\alpha' - \delta\alpha = 0$$

and the plane of two complanar lines is $V\alpha\beta + \omega S\alpha'\beta$.

If $P = \delta + \omega\rho$ is a point, and $\Pi = \rho' + \omega\delta'$ is a plane, P is in Π if

$$\delta\delta' - S\rho\rho' = 0.$$

Now consider the plane

$$\Pi = V\rho\alpha - \delta\alpha' - \omega S\rho\alpha'.$$

We can verify at once that this plane passes through P and that if $\alpha + \omega\alpha'$ is a line Π also passes through the line. That is, if $A = P + \omega\alpha'$ is a line the plane Π is the plane (PA). If A is not a line Π is said to be conjugate to WP with respect to the screw A.

Let $\rho = (\alpha\beta\gamma)$: let $\Pi = (\alpha'\beta'\gamma') + \omega\delta'$: let $\alpha = (fgh)$, $\alpha' = -(abc)$, we get

$$(\alpha'\beta'\gamma'\delta') = \begin{pmatrix} 0 & h & -g & a \\ -h & 0 & f & b \\ g & -f & 0 & c \\ -a & -b & -c & 0 \end{pmatrix}(\alpha\beta\gamma\delta).$$

If $\Pi\Pi'$ are two planes, their line of intersection is the axis of the cylindroid (Π, Π'): that is, if $\Pi = \varpi + \omega\varepsilon$, $\Pi' = \varpi' + \omega\varepsilon'$, their intersection is

$$-V\varpi\varpi' + \omega(\varpi\varepsilon' - \varpi'\varepsilon).$$

To show that this agrees with the former definition of a line I prove that if P, P' are points and both are in both of the planes $\Pi\Pi'$, that the line joining PP' is the same as the line of intersection of Π, Π'.

We have if $P = \delta + \omega\rho$, $P' = \delta' + \omega\rho'$

$$S\rho\varpi - \delta\varepsilon = 0$$
$$S\rho'\varpi - \delta'\varepsilon = 0$$
$$S\rho\varpi' - \delta\varepsilon' = 0$$
$$S\rho'\varpi' - \delta'\varepsilon' = 0.$$

The first two equations give, if we eliminate ε

$$\begin{aligned} 0 &= \delta' S\rho\varpi - \delta S\rho'\varpi \\ &= S\varpi(\rho\delta' - \rho'\delta). \end{aligned}$$

Similarly the third and fourth give

$$0 = S\varpi'(\rho\delta' - \rho'\delta).$$

Therefore $$\rho\delta' - \rho'\delta = xV\varpi\varpi',$$

where x is a scalar.

Operating with $V\rho$ we get

$$\begin{aligned} -\delta V\rho\rho' &= xV.\rho V\varpi\varpi' \\ &= x(\varpi' S\rho\varpi - \varpi S\rho\varpi') \\ &= \delta x(\varpi'\varepsilon - \varepsilon'\varpi) \end{aligned}$$

or $$V\rho\rho' = x(\varpi\varepsilon' - \varpi'\varepsilon).$$

Therefore $$\frac{\rho\delta' - \rho'\delta}{V\varpi\varpi'} = \frac{V\rho\rho'}{\varpi\varepsilon' - \varpi'\varepsilon},$$

which proves the theorem.

It is obvious that $$\mho(Q.\omega Q') = e^2\Omega QQ'$$
$$\Omega(Q.\omega Q') = \mho QQ'.$$

Therefore $$\mho S(Q.K\omega Q') = e^2\Omega S(QKQ')$$
$$\Omega S(Q.K\omega Q') = \mho S(QKQ')$$
$$T(\omega Q) = eTQ.$$

These equations give $$\cos(Q.\omega Q') = \sin e\,[QQ']$$
$$\sin e\,[Q.\omega Q'] = \cos(QQ').$$

Now let Q, Q' represent lines: then since $[QQ'] = 0$ is the condition that QQ' may intersect we can say that Q is at right angles to Q if it meets ωQ: or calling $\omega Q'$ the conjugate of Q' we can say that if one line cuts another at right angles, it cuts that line and its conjugate.

10. *Screws.*

We have seen that a biscalar represents a screw, and that a bivector represents a line: therefore the axis of a screw Q is a line R such that

$$Q = AR$$

where A is a biscalar.

Let Q' be another screw, R' its axis, and let $Q' = A'R'$. Then

$$SQKQ' = AA'SRKR'.$$

Therefore if $SRKR' = 0$, $SQKQ' = 0$, and conversely, unless AA' vanishes: but it can be easily verified that AA' cannot vanish unless QQ' are lines.

Therefore if the axes of two screws cut at right angles we have $(QQ') = \frac{\pi}{2}$, $[QQ'] = 0$, and conversely.

Therefore if we denote the axis of the cylindroid (QQ') by R we see at once (since $S(Q.VQKQ') = 0$) that the axes of all screws of the cylindroid cut the axis of R at right angles.

It is worth while to show that the expression for the axis of a screw agrees with the known formulæ for parabolic space.

We have, if $Q = \alpha + \omega\alpha'$

$$\operatorname{ec}\phi R = \left(\operatorname{ec}\frac{\varphi}{2} - \omega \operatorname{es}\frac{\varphi}{2}\right)(\alpha + \omega\alpha')$$

$$= \left(\alpha . \operatorname{ec}\frac{\varphi}{2} - e^2\alpha' \operatorname{es}\frac{\varphi}{2}\right) + \omega\left(\alpha' \operatorname{ec}\frac{\varphi}{2} - \alpha \operatorname{es}\frac{\varphi}{2}\right).$$

Now put $e = 0$: we get $\qquad \operatorname{ec}\frac{\varphi}{2} = 1 : \operatorname{es}\frac{\varphi}{2} = \frac{\varphi}{2}$

$$R = \alpha + \omega\left(\alpha' - \alpha \cdot \frac{\varphi}{2}\right).$$

Now $\qquad \frac{\varphi}{2} = \frac{-2S\alpha\alpha'}{T^2\alpha}.$

Therefore $\qquad R = \alpha + \omega\left(\alpha' - \frac{S\alpha\alpha'}{T^2\alpha}\alpha\right),$

which agrees with the quaternion expression I have given in Vol. XII of the Messenger of Mathematics, p. 130, if we write α for α' and $-\beta$ for α.

11. *The Cylindroid.*

The word *cylindroid* is used in two senses: it means either the set $\lambda Q + \mu Q'$, or taking Q, Q' to be bivectors, so that their axes are lines, it means the surface which is the locus of the axes of the screws of the set $\lambda Q + \mu Q'$.

I proceed to find the equation of the cylindroid.

I premise that, as is easily proved, the six edges of the tetrahedron of reference are $i, j, k, \omega i, \omega j, \omega k$.

The cylindroid contains two screws whose axes intersect at right angles: take these as the screws defining the cylindroid, and their axes as edges of the tetrahedron of reference.

Let the screws be
$$A = fi + a\omega i$$
$$B = gj + b\omega j.$$

Let $X = \lambda A + \mu B$ be any screw of the cylindroid: let $Y = xX + y\omega X$ be its axis. Then
$$X = \lambda fi + \mu gj + \omega(\lambda ai + \mu bj)$$
$$Y = \lambda i(fx + e^2 ay) + \mu j(gx + e^2 by) + \omega\{\lambda i(ax + fy) + \mu j(bx + gy)\}.$$
Therefore the coordinates of Y are
$$-\lambda(ax + fy),\ -\mu(bx + gy),\ 0,\ \lambda(xf + e^2 ay),\ -\mu(xg + e^2 by),\ 0.$$
Therefore if the point $(\alpha\beta\gamma\delta)$ is on the axis we must have
$$\begin{aligned}
&\text{(i)}\ \lambda\alpha(xa + yf) + \mu\beta(xb + yg) = 0\\
&\text{(ii)}\ \lambda\delta(xa + yf) - \mu\gamma(xg + e^2 by) = 0\\
&\text{(iii)}\ \lambda\gamma(xf + e^2 ay) + \mu\delta(xb + yg) = 0\\
&\text{(iv)}\ \lambda\beta(xf + e^2 ay) - \mu\alpha(xg + e^2 by) = 0.
\end{aligned}$$
We have to eliminate λ, μ, x, y.

Eliminating λ, μ, first between (i) (ii) and then between (i) (iii) we get, after a rearrangement of the terms,
$$x(g\alpha\gamma + b\beta\delta) + y(e^2 b\alpha\gamma + g\beta\delta) = 0$$
$$x(a\alpha\delta - f\beta\gamma) + y(f\alpha\delta - e^2 a\beta\gamma) = 0.$$

Eliminating x, y we get the equation of the cylindroid in the form
$$\begin{aligned}
0 &= (g\alpha\gamma + b\beta\delta)(f\alpha\delta - e^2 a\beta\gamma) - (a\alpha\delta - f\beta\gamma)(e^2 b\alpha\gamma + g\beta\delta)\\
&= (\alpha^2 + \beta^2)\gamma\delta(fg - e^2 ab) - \alpha\beta(e^2\gamma^2 + \delta^2)(ag - bf).
\end{aligned}$$
Now let ϕ, ϕ' be the pitches of the screws
$$\sin e\phi = \frac{2eaf}{a^2e^2 + f^2}$$
$$\sin\frac{e\varphi}{2} = \frac{ea}{\sqrt{(e^2a^2 + f^2)}}$$
$$\cos\frac{e\varphi}{2} = \frac{f}{\sqrt{(e^2a^2 + f^2)}}.$$
Therefore the equation of the cylindroid is
$$\mathrm{ec}\,\tfrac{1}{2}(\phi + \phi')(\alpha^2 + \beta^2)\gamma\delta - \mathrm{es}\,\tfrac{1}{2}(\phi - \phi')(e^2\gamma^2 + \delta^2)\alpha\beta = 0.$$
For parabolic space this is
$$(\alpha^2 + \beta^2)\gamma\delta - \frac{\varphi - \varphi'}{2}\delta^2\alpha\beta = 0.$$
Thus there is a factor δ, and putting $\delta = 1$ in the other factor we get
$$\gamma(\alpha^2 + \beta^2) - \frac{\varphi - \varphi'}{2}\alpha\beta = 0.$$

But it is obvious from the expression for ϕ in the last section that ϕ is double what is generally called the pitch, therefore this last equation agrees with the known form for parabolic space.

Writing
$$\alpha = \beta \tan \vartheta$$
$$\gamma = \delta \operatorname{et} \psi$$
$$\left(\operatorname{et} \psi = \frac{\tan e\psi}{e}\right)$$
we get from the equation of the cylindroid
$$\operatorname{es} 2\psi = \frac{\operatorname{es} \frac{1}{2}(\varphi - \varphi')}{\operatorname{ec} \frac{1}{2}(\varphi + \varphi')} \sin 2\vartheta.$$
Now we have
$$X = \lambda A + \mu B.$$
Therefore
$$SAKX = \lambda NA.$$
$$NX = \lambda^2 NA. + \mu^2 NB.$$
$$SBKX = \mu NB.$$
Therefore
$$\cos(AX) = \frac{\lambda}{\sqrt{\lambda^2 + \mu^2}}$$
$$\sin(AX) = \frac{\mu}{\sqrt{\lambda^2 + \mu^2}}.$$
Let $(AX) = l$. Then $\operatorname{es}[X] = \cos^2 l \operatorname{es}[A] + \sin^2 l \operatorname{es}[B]$.

It can be shown that we have
$$\operatorname{et} 2\psi = \frac{\sin 2l . \operatorname{es} \frac{1}{2}(\varphi - \varphi')}{\cos^2 l \operatorname{ec} \varphi + \sin^2 l \operatorname{ec} \varphi'}.$$

It is hardly worth while to stop to point out how these formulæ agree with the known formulæ of the theory of screws.

Prof. Ball has* shown that the cylindroid can be represented in a plane by means of a circle: the relation he uses is
$$(p - p_0)^2 + z^2 = m^2,$$
where p is the pitch of any screw of a cylindroid: z is the intercept its axis makes on the axis of the cylindroid, and, P_α, P_β being the pitches of the two screws of reference,
$$P_0 = \tfrac{1}{2}(P_\alpha + P_\beta);\ m = \tfrac{1}{2}(P_\alpha - P_\beta).$$

The general formula is, if χ is the pitch of any screw of the cylindroid, and ϕ, ϕ', ψ have the same meanings as before
$$4(\operatorname{es}\phi - \operatorname{es}\chi)(\operatorname{es}\chi - \operatorname{es}\phi') \operatorname{es}^2 \frac{\varphi - \varphi'}{2}$$
$$= \operatorname{et}^2 2\psi \{\operatorname{ec}\phi(\operatorname{es}\phi - \operatorname{es}\chi) + \operatorname{ec}\phi'(\operatorname{es}\chi - \operatorname{es}\phi)\}^2.$$

* Ball. On a plane dynamical representation of certain dynamical problems in the theory of a rigid body. 4 Proc. R. I. A., 2nd S. 29.

This equation is got by combining the value of et 2ψ with the expression for the pitch of any screw in terms of l: it is not hard to show that this reduces to Prof. Ball's equation if we take $e = 0$.

12. *Vectors.*

Clifford says that the vector of either species answers to Hamilton's vector: it is worth while to see how this comes about.

Let $\alpha + \omega\alpha'$ be a bivector, and suppose it is to be an η-vector: we must have if $\alpha = (fgh)$, $\alpha' = (abc)$

$$fi + gj + hk + \omega(ai + bj + ck) = e(ai + bj + ck) + \omega e^{-1}(fi + gj + ck).$$

Therefore
$$f = ae$$
$$g = be$$
$$h = ce.$$

Therefore in parabolic space, that is for $e = 0$, we get $f = g = h = 0$: that is, in parabolic space the vector is of the form $\omega\alpha$, which is a more precise form of Clifford's statement.

13. *Parallels.*

In this section we shall have to make more explicit use of the absolute than in what precedes, and the methods used will be mixed: partly biquaternions, partly geometricals, and partly algebraic.

We must bear in mind the stipulations in (7), especially the third.

Any straight line meets four generators of the absolute: for it meets the absolute in two points, and there are two generators through each point.

A generator is either a ξ-vector or an η-vector (7); call these two systems ξ-generators and η-generators respectively. Two lines are parallel if they meet the same two generators of the absolute: there are two cases: the generators may be of the same species or they may be of different species. In the former case the lines are said to be α-parallel: in the latter case they are said to be β-parallel. The conception of α-parallel lines is due to Clifford and Lindemann:* β-parallels are what is generally known as parallels. I proceed to find the conditions for parallelism.

The coordinates of any ξ- or η-vector are of the form

$$(abc \pm ae, \ \pm be, \ \pm ce),$$

* Clifford. Preliminary Sketch. Lindemann, 7 Math. Ann.

Moreover, if the vector is a line, we must have

$$af + bg + ch = 0,$$

and this gives $$a^2 + b^2 + c^2 = 0.$$

We can therefore take $a : b : c = i : \cos\vartheta : \sin\vartheta$,

and the coordinates of any generator will be

$$(i, \cos\vartheta, \sin\vartheta, \pm ei, \pm e\cos\vartheta, \pm e\sin\vartheta).$$

Now we must stop to consider the distinction into species.

If $\alpha + \omega\alpha'$ is an η-vector we have

$$\alpha + \omega\alpha' = e^{-1}\omega(\alpha + \omega\alpha')$$
$$= e\alpha' + e^{-1}\omega\alpha.$$

That is $\alpha = e\alpha'$: now if $(abcfgh)$ are the coordinates of the vector we have $\alpha = (fgh)$, $\alpha' = -(abc)$: therefore for an η-vector we have $f = -ea$, etc.: therefore the coordinates of an η-vector are

$$(abc - ea - eb, - ec)$$

and the coordinates of a ξ-vector are

$$(abc \;\; ea \;\; eb \;\; ec).$$

Moreover

$$2\,{}^{\xi}_{\eta}\,(\alpha + \omega\alpha') = (f \mp ea,\ g \mp eb,\ h \mp ec,\ -a \pm e^{-1}f,\ -b \pm e^{-1}g,\ -c \pm e^{-1}h).$$

Now let $(abcfgh)$ be any line: if it is to meet the ${}^{\xi}_{\eta}$ generator ϑ, we must have
$$0 = fi + g\cos\vartheta + h\sin\vartheta \pm eai \pm eb\cos\vartheta \pm ec\sin\vartheta$$
$$= i(f \pm ea) + (g \pm eb)\cos\vartheta + (h \pm ec)\sin\vartheta.$$

That is the line $A = (abcfgh)$ meets the ξ-generators, for which ϑ has the values determined by $i(f + ae) + \cos\vartheta(g + eb) + \sin\vartheta(h + ec) = 0$.

Similarly $A' = (a'b'c'f'g'h')$ meets the ξ-generators, for which ϑ has the values determined by $i(f' + a'e) + \cos\vartheta(g' + eb') + \sin\vartheta(h' + ec') = 0$.

Therefore if A, A' are to meet the same two ξ-generators these two equations must coincide, and we get

$$\frac{ae + f}{a'e + f'} = \frac{be + g}{b'e + g'} = \frac{ce + h}{c'e + h'} = \lambda.$$

That is $$\eta A = \lambda\eta A'.$$

That is A, A' are η-parallel.

In the same way if A, A' meet the same two η-generators they are ξ-parallel.

It follows, from what was proved in (5), that parallel lines meet the same generators: we have now proved the converse of this.

What we have proved is that α-parallelism, as now defined, is the same as what we called parallelism in (5).

I now consider the condition for β-parallelism: this is known to be that the lines intersect and that their angle vanishes: I show how this comes out from the definition given above, viz. that the two lines meet the same ξ-generator and also the same η-generator. Take the equation determining the ξ-generator cut by a line, viz. $i(f+ae)+\cos\vartheta\,(g+eb)+\sin\vartheta\,(h+ec)=0$.

Write

$$2\cos\vartheta = x+\frac{1}{x}$$

$$2\sin\vartheta = \frac{1}{i}\left(x-\frac{1}{x}\right)$$

we get

$$-2x(f+ae)+i(x^2+1)(g+eb)+(h+ec)(x^2-1)=0,$$

or

$$x^2\{i(g+eb)+(h+ec)\}-2x(f+ae)+\{i(g+eb)-(h+ec)\}=0.$$

Say this is

$$Ax^2 + 2Bx + C = 0.$$

In the same way the line $(a'b'c'f'g'h')$ gives an equation

$$A'x^2 + 2B'x + C' = 0.$$

If the lines are β-parallel the resultant of these equations must vanish: that is we must have

$$(AC'+A'C-2BB')^2 = 4(AC-B^2)(A'C'-B'^2).$$

We must remember that

$$0 = af+bg+ch = a'f'+b'g'+c'h'.$$

We have

$$\begin{aligned} AC-B^2 &= \{i(g+eb)+(h+ec)\}\{i(g+eb)-(h+ec)\}-(f+ae)^2 \\ &= -(g+eb)^2-(h+ec)^2-(f+ae)^2 \\ &= -(f^2+g^2+h^2+a^2e^2+b^2e^2+c^2e^2) \\ &= -X^2, \text{ say.} \end{aligned}$$

Similarly

$$\begin{aligned} A'C'-B'^2 &= -(f'^2+g'^2+h'^2+a'^2e^2+b'^2e^2+c'^2e^2) \\ &= -X'^2. \end{aligned}$$

$$\begin{aligned} AC'+A'C-2BB' &= \{i(g+eb)+(h+ec)\}\{i(g'+eb')-(h'+ec')\} \\ &\quad +\{i(g+eb)-(h+ec)\}\{i(g'+eb')+(h'+ec')\}-2(f+ae)(f'+a'e) \\ &= -2(g+eb)(g'+eb')+2(h+ec)(h'+ec') \\ &\qquad -2(f+ae)(f'+a'e) \\ &= -2(ff'+gg'+hh'+aa'e^2+bb'e^2+cc'e^2) \\ &\quad -2e(ch'+c'h+bg'+b'g+af'+a'f) \\ &= -2Y-2eZ, \text{ say.} \end{aligned}$$

Therefore we must have

$$(Y+eZ)^2 = X^2X'^2$$

or

$$Y+eZ = XX'.$$

But the two lines must also meet the same η-generator: this gives another equation which is obviously got by writing $-e$ for e: therefore we must have

$$Y + eZ = XX'$$
$$Y - eZ = XX'.$$

Therefore we must have
$$Z = 0$$
$$Y = XX'.$$

But $Z = 0$ gives $\quad af' + a'f + bg' + b'g + ch' + c'h = 0,$

that is to say, $Z = 0$ is the condition that the lines may intersect: and $Y = XX'$ gives

$$1 = \frac{ff' + gg' + hh' + e^2(aa' + bb' + cc')}{(f^2 + g^2 + h^2 + e^2a^2 + e^2b^2 + e^2c^2)^{\frac{1}{2}}.(f'^2 + g'^2 + h'^2 + e^2a'^2 + e^2b'^2 + e^2c'^2)^{\frac{1}{2}}} = \cos\phi,$$

if ϕ is the angle between the lines.

Therefore if two lines are β-parallel they intersect, and the angle between them vanishes. It is worth noticing that in parabolic space the condition $Z = 0$ disappears, since both resultants only give

$$Y = XX'.$$

Moreover, the condition for α-parallelism is

$$\frac{f}{f'} = \frac{g}{g'} = \frac{h}{h'},$$

which gives $\cos\phi = 1$: so that the conditions for the two species of parallelism coincide, as of course they should do.

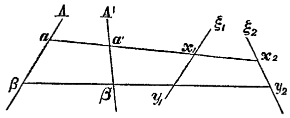
The investigations which follow are mostly developments of (4) on p. 192 of Clifford's papers, and of Note (i) on p. 642.

Through any point we can draw two lines parallel to a given line, viz. they are the two lines drawn through the point, one cutting the two η-generators cut by the line, and the others cutting the two ξ-generators cut by the line. To find the locus of parallels to a given line drawn through the points of a given line. Let the given lines be A, B: suppose the lines are to be drawn η-parallel to A: then they must meet the two ξ-generators cut by A: but they also meet B: therefore they are one system of generators of a quadric and B and the two ξ-generators are three lines of the other system.

Now B meets two η-generators: these also cut the two ξ-generators: therefore the quadric contains two ξ-generators and two η-generators: all its generators of the one system cut B and the two ξ-generators, and all its generators of the other system cut A and the two η-generators.

Therefore one system of generators of the new quadric (call it the quadric Σ) consists of lines through B, η-parallel to A, and the other system consists of lines through A, ξ-parallel to B.

Let ξ_1, ξ_2, η_1, η_2 be the two ξ-generators, and the two η-generators of Σ: let A, A' be two generators of Σ, of the same system as ξ_1, ξ_2: let B, B' be two



other generators, cutting $AA'\xi_1\xi_2$ in α, α', x_1, x_2, β, β', y_1, y_2, respectively: it need hardly be remarked that the figure is not supposed to represent the actual state of things, but only to show on what lines the points are supposed to lie.

We have by the fundamental properties of a ruled quadric

$$\{\alpha\alpha' x_1 x_2\} = \{\beta\beta' y_1 y_2\}$$

if the $\{\,\}$ denote anharmonic ratios: but x_1, x_2, y_1, y_2 are the points in which $\alpha\alpha'$, $\beta\beta'$ cut the absolute. Therefore, remembering the anharmonic ratio definition of distance, we get

$$\alpha\alpha' = \beta\beta'.$$

That is to say, any two parallels of one system cut off equal intercepts on two fixed parallels of the other system.

Now consider the equation proved in (5), viz.

$$\frac{\cos(Qu) + \sin e\,[Qu]}{(1 + \sin e\,[Q])^{\frac{1}{2}}.(1 + \sin e\,[u])^{\frac{1}{2}}} = \frac{\cos(RW) + \sin e\,[RW]}{(1 + \sin e\,[R])^{\frac{1}{2}}.(1 + \sin e\,[W])^{\frac{1}{2}}}.$$

Let $UVWR$ be lines, and let Q, U intersect, and also R, W: then we have

$$0 = [U] = [V] = [W] = [R] = [QU] = [RW]$$

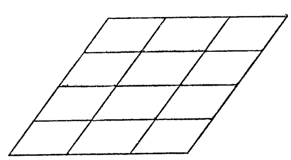
and we get

$$\cos(QU) = \cos(RW).$$

Then we can take $U = W$, and we get

$$\cos(QU) = \cos(RU).$$

That is to say, if two lines intersect, they make the same angle as any two parallels to them that intersect, and in particular, any line meeting two parallels makes equal angles with them. We see from all this that the geometry of the surface Σ is the same as the geometry of a parallelogram: for in a parallelogram we have two parallelisms, and every line of one parallelism makes equal angles with every line of the other parallelism, and two fixed lines of one parallelism make a constant intercept on all lines of the other parallelism: and all this is true for the generators of the surface Σ.



I shall now find the equation to the surface Σ, referred to the rectangular system used in this paper.

Let the line through which the parallels are to be drawn be the intersection of the planes $lx + my + nz + p\omega = 0$, $l'x + m'y + n'z + p'\omega = 0$.

Let $x'y'z'\omega'$ be a point on this line: $(xyz\omega)$ a point on the required locus: then, since the line joining the two points is parallel to a fixed line, and since $(x'y'z'\omega')$ is on the given line, we have a set of equations

$$\begin{aligned}
x'\omega + ey'z - ez'y - \omega'x &= \lambda\alpha \\
-ex'z + y'\omega + ez'x - \omega'y &= \lambda\beta \\
ex'y - ey'x + z'\omega - \omega'z &= \lambda\gamma \\
lx' + my' + nz' + p\omega' &= 0 \\
l'x' + m'y' + n'z' + p'\omega' &= 0.
\end{aligned}$$

Eliminating λ, x', y', z', ω' and writing $abcfgh$ for the coordinates of the intersection of the two planes we get

$$\begin{vmatrix}
\omega & ez & -ey & -x\alpha \\
-ez & \omega & ex & -y\beta \\
ey & -ex & \omega & -z\gamma \\
l & m & n & p0 \\
l' & m' & n' & p'0
\end{vmatrix} = 0.$$

or $\alpha\,[a(e^2x^2 + \omega^2) + ef(y^2 + z^2) + (ec - h)(ezx - y\omega) + (eb - g)(exy + z\omega)]$
$+ \beta\,[b(e^2y^2 + \omega^2) + eg(z^2 + x^2) + (ea - f)(exy - z\omega) + (ec - h)(eyz + x\omega)]$
$+ \gamma\,[c(e^2z^2 + \omega^2) + eh(x^2 + y^2) + (eb - g)(eyz - x\omega) + (ea - f)(ezx + y\omega)]$
$= 0$.

It is worth while to see how this quadric reduces to a plane in parabolic space: we have to put $e = 0$, and then there is a factor ω, and putting $\omega = 1$ in the remaining factor, we get

$$0 = \alpha(a + hy - gz) + \beta(b + fz - hx) + \gamma(c + gx - fy).$$

In this form we see that the plane contains the given line $(abcfgh)$, and writing it in the form

$$0 = \alpha a + \beta b + \gamma c + x(g\gamma - h\beta) + y(h\alpha - f\gamma) + z(f\beta - g\alpha)$$

we see that it is parallel to $(\alpha\beta\gamma)$, as, of course, it should be.

I now change the axes of reference, and take

$$S = x\omega - yz = 0$$

as the equation of the absolute, and

$$\Sigma = x\omega - \lambda yz = 0$$

as the equation of the surface Σ: so that the four common generators are

$$(xy),\ (xz),\ (\omega y),\ (\omega z).$$

The polar of $(x''y''z''\omega'')$ with respect to S is

$$x\omega'' - yz'' - zy'' + \omega x'' = 0.$$

Therefore the pole of $(lmnp)$ is $(p, -n, -m, l)$.

The tangent plane to Σ at $(x'y'z'\omega')$ is

$$x\omega' - \lambda yz' - \lambda zy' + \omega x'.$$

Therefore its pole with respect to S is $(x', \lambda y', \lambda z', \omega')$, and the line joining this pole to $(x'y'z'\omega')$, that is the normal at $(x'y'z'\omega')$ has for its coordinates the minors of

$$\left\| \begin{matrix} x' & y' & z' & \omega' \\ x' & \lambda y' & \lambda z' & \omega' \end{matrix} \right\|,$$

or $$\big(0,\ xz(1-\lambda),\ xy(\lambda-1),\ 0,\ y\omega(1-\lambda),\ z\omega(1-\lambda)\big),$$

or dividing by $(1-\lambda)$ $$(0,\ xz,\ -xy,\ 0,\ y\omega,\ z\omega).$$

Therefore all the normals meet yz and $x\omega$; therefore through any point not on the surface we can only draw one normal (instead of *six*) to the surface: this normal is the line through the point cutting yz, $x\omega$.

The coordinates of the generators of Σ can be easily proved to be of the form $$\rho,\ -\rho^2,\ 0,\ \lambda\rho,\ \lambda,\ 0)$$

for the one system: call this the η-system. And

$$(-\rho,\ 0,\ \rho^2,\ \lambda\rho,\ 0,\ \lambda)$$

for the other system: call this the ξ-system.

For the generators of S we have, therefore,

$$(\rho,\ -\rho^2,\ 0,\ \rho,\ 1,\ 0)$$

and $$(-\rho,\ 0,\ \rho^2,\ \rho,\ 0,\ 1).$$

Therefore a line meets the η-generators determined by

$$f\rho - g\rho^2 + a\rho + b = 0,$$

or $$g\rho^2 - (a+f)\rho - b = 0.$$

Therefore two lines are ξ-parallel if

$$\frac{a+f}{a'+f'} = \frac{b}{b'} = \frac{g}{g'}.$$

In the same way two lines are η-parallel if

$$\frac{a-f}{a'-f'} = \frac{c}{c'} = \frac{h}{h'}.$$

The normal at a point $(xyz\omega)$ of Σ was found to be

$$(0,\ xz,\ -xy,\ 0,\ y\omega,\ z\omega).$$

Therefore the normals at $(xyz\omega)$, $(x'y'z'\omega')$ are ξ-parallel if

$$\frac{xz}{x'z'} = \frac{y\omega}{y'\omega'}. \tag{A}$$

They are η-parallel if $$\frac{xy}{x'y'} = \frac{z\omega}{z'\omega'}. \tag{B}$$

Now take equation (A): this asserts that the normals at all points of the intersection of (Σ) $x\omega - \lambda yz$ with $y\omega - \mu zx$ are parallel: and in the same way the normals at all points of the intersection of $x\omega - \lambda yz$ with $z\omega - \nu xy$ are parallel: now I say that all these intersections are straight lines, and moreover that the systems $(x\omega - \lambda yz,\ y\omega - \mu zx)$ are orthogonal, as also the systems $(x\omega - \lambda yz,\ z\omega - \nu xy)$, and that at every point of $x\omega - \lambda yz$ the other two surfaces cut at a constant angle.

Consider the surface $y\omega - \mu zx = 0$: we have

$$\mu = \frac{y\omega}{zx}.$$

But we have also at any point of Σ, $x\omega - \lambda yz, = 0$, or

$$\frac{\omega}{z} = \lambda \frac{y}{x}.$$

Therefore at any point of the intersection of the two surfaces we have

$$\mu = \lambda \frac{y^2}{x^2}.$$

Therefore the surface $\mu =$ const. cuts the surface $\lambda =$ const. in two generators determined by

$$\frac{y}{x} = \pm \sqrt{\frac{\mu}{\lambda}}.$$

Moreover the surfaces have the lines (xy), $(z\omega)$ in common: therefore $\mu =$ const. cuts $\lambda =$ const. in four straight lines: in the same way we can show that $\nu =$ const. cuts $\lambda =$ const. in four straight lines, two being (zx), $(y\omega)$, and the other two given by

$$\frac{z}{x} = \pm \sqrt{\frac{\nu}{\lambda}}.$$

Now to show that $\mu =$ const., $\lambda =$ const. are orthogonal. The tangent planes at $x'y'z'\omega'$ are

$$x\omega' - \lambda yz' - \lambda zy' + \omega x' = 0$$
$$-\mu xz' + y\omega' - \mu zx' + \omega y' = 0.$$

Now the plane equation of the absolute is $lp - mn = 0$: therefore two planes $(lmnp)(l'm'n'p')$ are at right angles if

$$lp' + l'p - mn' - m'n = 0.$$

In the present case this is

$$\begin{aligned} 0 &= \omega'y' - \mu x'z' - \lambda\mu x'z' + \lambda y'\omega' \\ &= (\omega'y' - \mu x'z')(1 + \lambda), \end{aligned}$$

which is right, since $\omega'y' - \mu x'z' = 0$.

We can say more generally, that if we take any point on $\mu =$ const. its tangent plane is at right angles to its polar with respect to $\lambda =$ const.

The same is obviously true for the surface $\nu =$ const.

Now consider the two surfaces

$$y\omega - \mu zx = 0$$
$$z\omega - \nu xy = 0.$$

Their tangent planes are

$$-\mu xz' + y\omega' - \mu zx' + \omega y' = 0$$
$$-\nu xy' - \nu yx' + z\omega' + \omega z' = 0.$$

Therefore the angle between them is given by

$$\cos\vartheta = \frac{1}{2} \cdot \frac{-\mu z'^2 - \nu y'^2 - \omega'^2 - \mu\nu x'^2}{\sqrt{(\mu x'\omega' - \mu y'z')(\nu x'\omega' - \nu y'z')}}$$
$$= -\frac{1}{2\sqrt{\mu\nu}} \cdot \frac{\mu\nu x'^2 + \nu y'^2 + \mu z'^2 + \omega'^2}{(x'\omega' - y'z')}.$$

Now, leaving out accents, we have

$$\mu = \frac{y\omega}{xz}$$
$$\nu = \frac{z\omega}{xy},$$

and therefore

$$\mu\nu = \frac{\omega^2}{x^2}.$$

Therefore the numerator of $\cos\vartheta$ becomes

$$\omega^2 + \frac{yz\omega}{x} + \frac{yz\omega}{x} + \omega^2$$
$$= \frac{2\omega}{x}(x\omega + yz)$$
$$= 2\sqrt{\mu\nu}\,(x\omega + yz).$$

Therefore

$$\cos\vartheta = \frac{yz + x\omega}{yz - x\omega}$$
$$= \frac{1+\lambda}{1-\lambda},$$

if $x\omega = \lambda yz$ is the surface of the λ-system passing through the point: therefore all we can say is that for all the intersections of three surfaces, $\lambda =$ const., $\mu =$ const., $\nu =$ const., the tangent planes to the last two cut at the same angle.

There is another way of treating the theory of parallels, which should be noticed.

If we take any two lines, Q, R, there are in general two lines meeting them both at right angles: for it is obvious from what was proved at the end of (9) that these are the four lines meeting Q, R, ωQ, ωR: it may, however,

happen that there is an infinite number of such lines: if this is the case we have Q, R, ωQ, ωR connected by a linear relation of the form

$$aQ + b\omega Q + cR + d\omega R = 0.$$

And then it follows by what was proved in (5) that Q, R are parallel.

14. *Biquaternions.*

If Q is any biquaternion its reciprocal is defined by

$$Q^{-1} = \frac{KQ}{NQ},$$

where $\frac{1}{NQ} = (NQ)^{-1}$ as defined at the end of (1). We then get

$$\frac{\alpha|}{|\beta} = \alpha\beta^{-1},$$

so that α/β is a biquaternion.

Now taking α, β as two screws we have to see what the actual operation is which changes β into α: but this can obviously be done as follows: take the shortest distances of the axes of α, β, move an axis of β along them and make it coincide with an axis of α: then alter $N\beta$ until it is equal to $N\alpha$.

And this operation is a biquaternion: and we get an equation

$$(q + \omega q')(\alpha + \omega\alpha') = \beta + \omega\beta':$$

and we have $$S.V(q + \omega q')(\alpha + \omega\alpha') = 0.$$

We can say that a biquaternion Q can operate upon the bivector α if $S\alpha VQ = 0$: that is, if the axes of α cut the axes of Q at right angles. In this way, I think, we can get an explanation of a difficulty noticed by Clifford (p. 179).

Clifford gets the equation

$$(q + \omega r)(\alpha + \omega\beta) = \gamma + \omega\delta.$$

Then the difficulty is that the expression "$q + \omega r$ does not denote the sum of geometrical operations, which can be applied to the motor as a whole; and the ratio of two motors is only expressed by a symbol as the sum of two parts, each of which separately has a definite meaning in certain other cases, but not in the case in point."

Now I submit that we have no more right to break $npq + \omega r$ into its components q, r and expect each to operate upon $\alpha + \omega\beta$, than we should have to take the equation $q\alpha = \gamma$, to write q in the form

$$a + xi + yj + zk,$$

and to expect each of the three parts a, xi, yj, zk to operate separately upon α.

The cases are entirely analogous: a biquaternion $q + \omega r$, considered as an operator, is one and indivisible in exactly the same way as the quaternion $a + xi + yj + zk$ is one and indivisible when considered as an operator.

Clifford says that the difficulty does not occur in elliptic space: this is only because we can evade it by writing the biquaternion in the form $\xi q + \eta r$, and then we have $(\xi q + \eta r)(\xi\alpha + \eta\beta) = \xi(q\alpha) + \eta(r\beta)$ and both quaternions can operate if α, β are perpendicular to the axes of q, r respectively. If we say that two bivectors A, B are at right angles if $SAB = 0$, and if we call VQ (a bivector) for the moment the axis of RQ we can say that a biquaternion can operate upon any bivector at right angles to its axis, and changes it into another bivector at right angles to its axis.

In this sense, the axis of α/β is the axis of the cylindroid $(\alpha\beta)$, and just as in bivectors the cylindroid takes the place of the plane, so the axis of the cylindroid takes the place of the normal to the plane.

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